PRESCRIBING EIGENVALUES OF THE DIRAC OPERATOR

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Abstract. In this note we show that every compact spin manifold of dimension ≥ 3 can be given a Riemannian metric for which a finite part of the spectrum of the Dirac operator consists of arbitrarily prescribed eigenvalues with multiplicity 1.

1. Introduction and statement of results

The Dirac operator D is a formally self-adjoint first order elliptic differential operator acting on sections of the spinor bundle over a spin manifold. Standard elliptic theory tells us that if the manifold is compact and without boundary then the spectrum of the Dirac operator is a discrete subset of the real line. The relationship between the spectrum of D and topological and geometrical invariants of the manifold is a deep and interesting subject. Results of different types are known.

First and foremost there is the Index Theorem of Atiyah and Singer which gives a way of computing the Fredholm index of the Dirac operator by purely topological means. Another type of results are statements about the Dirac spectrum for generic Riemannian metrics. The Index Theorem gives a topological lower bound on the multiplicity of the zero eigenvalue of the Dirac operator, it is conjectured that this lower bound is sharp for generic Riemannian metrics.

Conjecture 1. *On any compact spin manifold for a generic Riemannian metric the space of harmonic spinors is not larger than it is forced to be by the index theorem.*

Conjecture 1 is known to be true for all manifolds of dimension ≤ 4 [11] and for dimension ≥ 5 for a large class of manifolds including all simply connected manifolds [2]. Also it is known that for a generic metric on a three-dimensional manifold it holds that all eigenvalues have multiplicity one [4]. It seems reasonable to believe that this holds true in any dimension.

In the present paper we will be concerned with the problem of finding specific metrics on a given spin manifold with "non-generic" properties of the Dirac spectrum. As a complement to Conjecture 1 there is the following.

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Conjecture 2. *On any compact spin manifold there is a Riemannian metric for which the kernel of the Dirac operator is non-trivial.*

From the work of Hitchin [7] and Bär [1] it is known that Conjecture 2 is true for spin manifolds of dimension $\equiv 0, 1, 3, 7 \pmod 8$, although exact information about the dimension of the kernel is not provided by their methods. In [12] Seeger constructs explicit metrics with non-trivial Dirac kernel on spheres of dimension $0 \pmod 4$.

These results can be formulated as saying that it is possible to prescribe zero to be an eigenvalue of the Dirac operator (with some unknown multiplicity) and then find a Riemannian metric for which this holds. We will show that it is possible to prescribe a finite part of the spectrum of the Dirac operator as eigenvalues with simple multiplicity, possibly with the exception of the zero eigenvalue. If Conjecture 1 is true we need not make this exception.

Since the spectrum of the Dirac operator is always symmetric about zero in dimensions $\not\equiv 3 \pmod 4$ we get a slightly different formulation of our main result depending on the dimension of the manifold. What we will prove is the following.

Theorem 3. *Let M be a compact spin manifold of dimension $n \geq 3$ and let $L > 0$ be a real number.*

- *Suppose that $n \equiv 3 \pmod 4$ and let $l_1, l_2, \dots, l_m$ be non-zero real numbers such that $-L < l_1 < l_2 < \cdots < l_m < L$. Then there is a Riemannian metric g on M such that* $\mathrm{spec}\,(\mathrm{D_g}) \cap ((-\mathrm{L}, \mathrm{L}) \setminus \{0\})$ *consists precisely of $l_1, l_2, \dots, l_m$ as simple eigenvalues.*
- *Suppose that $n \not\equiv 3 \pmod 4$ and let $l_1, l_2, \dots, l_m$ be real numbers such that $0 < l_1 < l_2 < \cdots < l_m < L$. Then there is a Riemannian metric g on M such that* $\mathrm{spec}\,(\mathrm{D_g}) \cap ((-\mathrm{L}, \mathrm{L}) \setminus \{0\})$ *consists precisely of $\pm l_1, \pm l_2, \dots, \pm l_m$ as simple eigenvalues.*

The idea of the proof is similar to the way Conjecture 2 is proved in [1]. Using the surgery result of [2] we begin by constructing metrics on spheres for which the spectrum D in a given interval consists of one simple eigenvalue. We can then start with any Riemannian spin manifold, rescale the metric and take a connected sum with finitely many spheres. The surgery theorem of [2] ensures that the resulting manifold, which is diffeomorphic to the original manifold, has a metric for which the Dirac eigenvalues in a given interval are precisely the simple eigenvalues on the spheres.

In [3] Colin de Verdier shows how to find metrics on compact manifolds for which a finite part of the spectrum of the Laplace operator acting on functions is arbitrarily prescribed. Lohkamp [10] refines this result by showing how to prescribe simultaneously a finite part of the Laplace spectrum, the volume, and certain curvature invariants.

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2. Preliminaries

A spin manifold is always assumed to be equipped with an orientation and a spin structure. If M is a spin manifold we denote the spinor bundle over M by ΣM.

There are two natural endomorphisms of the spinor representation, and thus of the fibre $\Sigma_x M$. Let $n = \dim M$. If n is even we consider Clifford multiplication by the volume form $\omega = (-1)^{n/2} e_1 \cdot e_2 \cdot \cdots \cdot e_n$. Multiplication by ω anti-commutes with multiplication by a single tangent vector.

Depending on the dimension n the complex spinor representation has either a real or a quaternionic structure, see for instance [5, sec. 1.7]. A real structure is an anti-linear endomorphism α with $\alpha^2 = \mathrm{Id}$, a quaternionic structure is an anti-linear endomorphism α with $\alpha^2 = -\mathrm{Id}$. If $n \equiv 0, 1, 6, 7 \pmod 8$ there is a real structure α on $\Sigma_x M$, if $n \equiv 2, 3, 4, 5 \pmod 8$ there is a quaternionic structure α on $\Sigma_x M$. The endomorphism α commutes with Clifford multiplication by tangent vectors if $n \equiv 2, 3, 6, 7 \pmod 8$ and anti-commutes otherwise. Both ω and α extend to parallel fields on M.

Proposition 4. *Suppose (M, g) is a compact Riemannian spin manifold of dimension $n \equiv 2, 3, 4 \pmod 8$. Then the eigenspaces of the Dirac operator are quaternionic vector spaces and thus have even complex dimension.*

Proof. If $n \equiv 2, 3 \pmod 8$ there is a quaternionic structure α which commutes with Clifford multiplication by tangent vectors. Since α is parallel it commutes with the Dirac operator so eigenspaces of D are quaternionic vector spaces.

If $n \equiv 4 \pmod 8$ then the composition $\omega \cdot \alpha$ is a parallel quaternionic structure which commutes with D since ω and α are both parallel and both anti-commute with Clifford multiplication. The eigenspaces of D are quaternionic vector spaces with respect to this quaternionic structure. □

When we talk of dimensions of eigenspaces of the Dirac operator we will always mean quaternionic dimension if $n \equiv 2, 3, 4 \pmod 8$ and complex dimension otherwise. We will call an eigenvalue of D "simple" if it is an eigenvalue for which the corresponding eigenspace is one-dimensional.

Proposition 5. *Suppose (M, g) is a compact Riemannian spin manifold of dimension $n \not\equiv 3, 7 \pmod 8$. Then the spectrum of the Dirac operator is symmetric about zero.*

Proof. Let φ be an eigenspinor of D with eigenvalue λ.

First suppose n is even. Then $\omega \cdot \varphi$ is an eigenspinor with eigenvalue $-\lambda$ since ω anti-commutes with D.

Next suppose $n \equiv 1, 5 \pmod 8$. Then the real/quaternionic structure α anti-commutes with D so $\alpha(\varphi)$ is an eigenspinor with eigenvalue $-\lambda$. □

The Index Theorem of Atiyah and Singer relates the dimension of the kernel of the Dirac operator to the topology of the manifold.

If n is even the spinor bundle splits as $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ where $\Sigma^\pm M$ are the ± 1-eigenbundles for multiplication by ω. The Dirac operator is a sum $D = D^+ \oplus D^-$ where $D^\pm$ maps sections of $\Sigma^\pm M$ to sections of $\Sigma^\mp M$. The Atiyah-Singer Index Theorem states that $\dim\ker D^+ - \dim\ker D^- = \widehat{A}(M)$ is a topological invariant, the $\widehat{A}$-genus of M. It follows that $\dim\ker D = \dim\ker D^+ + \dim\ker D^- \geq |\widehat{A}(M)|$. If $n \equiv 2 \pmod 4$ then $\widehat{A}(M) = 0$ and the only conclusion to be made is that $\ker D$ is even-dimensional.

In dimensions $\equiv 1, 2 \pmod 8$ the Index Theorem tells us that $\dim\ker D \equiv \alpha(M) \pmod 2$ and $\dim\ker D^+ \equiv \alpha(M) \pmod 2$ respectively, where $\alpha(M) \in \mathbb{Z}/2\mathbb{Z}$ is the topological α-genus. It follows that $\dim\ker D \geq |\alpha(M)|$ in dimensions $\equiv 1 \pmod 8$ and $\dim\ker D \geq 2|\alpha(M)|$ in dimensions $\equiv 2 \pmod 8$.

To measure closeness of the spectra of Dirac operators on different Riemannian manifolds we use the following definition. Let $\Lambda, \epsilon > 0$. Two operators with discrete real spectrum are called (Λ, ϵ)-spectral close if

- $\pm\Lambda$ are not eigenvalues of either operator,
- both operators have the same total number m of eigenvalues in the interval $(-\Lambda, \Lambda)$, and
- if the eigenvalues in $(-\Lambda, \Lambda)$ of the two operators are denoted by $\lambda_1 \leq \cdots \leq \lambda_m$ and $\mu_1 \leq \cdots \leq \mu_m$ respectively (each eigenvalue being repeated according to its multiplicity), then $|\lambda_i - \mu_i| < \epsilon$ for $i = 1, \ldots, m$.

We will consider the space of Riemannian metrics on a manifold equipped with the C^2-topology, when we speak of a continuous family of metrics it is continuous with respect to this topology. All arguments in the proof of Theorem 1.2 of [2] depend continuously on the Riemannian metric in the C^2-topology, and this gives the following generalization of that result.

Theorem 6. *Let M be a closed spin manifold equipped with a spin structure and a continuous family of Riemannian metrics g_x for x in some compact space X. Let $N \subset M$ be an embedded sphere of codimension ≥ 3 and with a trivialized tubular neighborhood. Let $\tilde{M}$ be the manifold with spin structure obtained from M by surgery along N.*

Let $\epsilon > 0$ and $\Lambda > 0$ such that $\pm\Lambda \notin \mathrm{spec}\,(\mathrm{D_{g_x}})$ for all x. Then there exists a family of Riemannian metrics $\tilde{g}_x$, $x \in X$, on $\tilde{M}$ such that D_{g_x} and $D_{\tilde{g}_x}$ are (Λ, ϵ)-spectral close for all $x \in X$.

3. Prescribing eigenvalues on spheres

In this section we will show how to construct Riemannian metrics on spheres with one Dirac eigenvalue prescribed in a given interval. The construction will use the surgery result of the previous section and some special Riemannian manifolds.

Proposition 7. *Let $n \geq 3$ be an integer and let l, L be real numbers with $L > 0$, $l \neq 0$, $l \in (-L, L)$.*

- *If* $n \equiv 3, 7 \pmod 8$ *there is a metric* $g^{l,L}$ *on* S^n *for which* $\operatorname{spec}\left(\mathrm{D}_{\mathrm{g}^{\mathrm{l,L}}}\right) \cap (-\mathrm{L}, \mathrm{L})$ *consists only of* l *as a simple eigenvalue.*
- *If* $n \not\equiv 3, 7 \pmod 8$ *there is a metric* $g^{l,L}$ *on* S^n *for which* $\operatorname{spec}\left(\mathrm{D}_{\mathrm{g}^{\mathrm{l,L}}}\right) \cap (-\mathrm{L}, \mathrm{L})$ *consists only of* $\pm l$ *as simple eigenvalues.*

To prove the proposition we first need to construct certain Riemannian manifolds with harmonic spinors. They are S^3 with the Berger metric and manifolds with special holonomy. In [7, sec. 3.1] Hitchin computes explicitly the Dirac spectrum of the family g_t^{B} of Berger metrics on the three-dimensional sphere S^3. In particular it is shown that for a certain parameter interval all eigenvalues except one are uniformly bounded away from zero, and the remaining simple eigenvalue changes sign when the parameter runs through the interval. For the existence of compact manifolds with special holonomy as required below we refer to [8].

We define the following manifolds to use as building blocks:

- (V^1, g^1), the 1-dimensional circle with the non-bounding spin structure. This has $\dim_{\mathbb{C}} \ker D_{g^1} = 1$.
- (V^3, g^3), where V^3 is the 3-sphere and $g^3 = g_{t_0}^{\mathrm{B}}$ is the Berger metric for the parameter value t_0 for which $\dim_{\mathbb{H}} \ker D_{g^3} = 1$.
- (V^4, g^4), a compact 4-manifold with holonomy $\mathrm{SU}(2)$. This has $\dim_{\mathbb{H}} \ker D_{g^4} = 1$, see [13, p. 59], [8, Thm. 3.6.1].
- (V^6, g^6), a compact 6-manifold with holonomy $\mathrm{SU}(3)$. Then $\dim_{\mathbb{C}} \ker D_{g^6} = 2$.
- (V^7, g^7), a compact 7-manifold with holonomy G_2. In this case we have $\dim_{\mathbb{C}} \ker D_{g^7} = 1$.
- (V^8, g^8), a compact 8-dimensional "Bott-manifold" with holonomy $\mathrm{Spin}(7)$. This has $\dim_{\mathbb{C}} \ker D_{g^8} = 1$.
- (V^{10}, g^{10}), a compact 10-manifold with holonomy $\mathrm{SU}(5)$. Then $\dim_{\mathbb{H}} \ker D_{g^{10}} = 1$.

The manifold (V^8, g^8) plays a special role, we are going to multiply with this manifold to increase dimensions. From [7, Rem. 4, p. 12] it follows that

$$\dim \ker D_{g+g^8} = \dim \ker D_g \tag{1}$$

for any manifold (V, g), where $g + g^8$ is the product metric on $V \times V^8$.

Lemma 8. *Suppose* $n \geq 3$.

- *For* $n \equiv 3, 7 \pmod 8$ *there is a manifold* (W^n, h^n) *such that* W^n *is a spin boundary and* $\dim \ker D_{h^n} = 1$.
- *For* $n \not\equiv 3, 7 \pmod 8$ *there is a manifold* (W^n, h^n) *such that* W^n *is a spin boundary and* $\dim \ker D_{h^n} = 2$.

The property that W^n is a spin boundary is equivalent to the sphere S^n being spin bordant to W^n. In the following we use the formulas in [7, Rem. 4, p. 12] to find the dimension of the kernel of the Dirac operator on a product manifold.

Proof of Lemma 8. If $n = 3 + 8p$ we set $W^n := V^3 \times (V^8)^p$ with h^n the product metric. Since $V^3 = S^3$ is a spin boundary we have that W^n is a spin boundary and from (1) it follows that $\dim \ker D_{h^n} = 1$.

If $n = 7 + 8p$ we set $W^n := V^7 \times (V^8)^p$ with the product metric. Any compact spin manifold of dimension 7 is a spin boundary [9, p. 92], so W^n is also a spin boundary. From (1) it follows that the product metric has $\dim \ker D_{h^n} = 1$.

If $n = 8p$ we take W^n as the disjoint union $(V^8)^p + \overline{(V^8)^p}$, where the overline means taking the opposite orientation. Then W^n is the boundary of the cylinder $(V^8)^p \times [0, 1]$, from (1) we have $\dim \ker D = 1$ on each component of W^n, so $\dim \ker D_{h^n} = 2$.

If $n = 1 + 8p$ we set $W^n := V^1 \times (V^8)^p + \overline{V^1 \times (V^8)^p}$, if $n = 2 + 8p$ we set $W^n := V^{10} \times (V^8)^{p-1} + \overline{V^{10} \times (V^8)^{p-1}}$, and if $n = 4 + 8p$ we set $W^n := V^4 \times (V^8)^p + \overline{V^4 \times (V^8)^p}$. In these cases W^n has the required properties for the same reasons as when $n = 8p$.

If $n = 5 + 8p$ we set $W^n := V^1 \times V^4 \times (V^8)^p$. Since V^4 has quaternionic eigenspaces and W^n has not we get $\dim \ker D_{h^n} = 2$ for the product metric. If $n = 6 + 8p$ we put $W^n := V^6 \times (V^8)^p$ equipped with the product metric. The product metric has $\dim \ker D_{h^n} = 2$. In dimensions $n = 5, 6$ all compact spin manifolds are boundaries [9, p. 92] so the manifolds W^n are spin boundaries as well. □

We now prove Proposition 7. The low dimensions $n = 3, 4$ must be treated separately since the general argument involves breaking down a bordism into elementary bordisms corresponding to surgeries of codimension at least three, and this requires $n \geq 5$.

Proof of Proposition 7. First assume $n = 3$. Rescale the Berger metrics g_t^{B} so that the interval $(-L, L)$ contains only one Dirac eigenvalue for t close to t_0. Choose the parameter t near t_0 to get a single (positive or negative) eigenvalue with absolute value less than $|l|$. Rescaling again increases this eigenvalue to l and we have a metric $g^{l,L}$ with the required properties.

Next assume $n = 4$. The manifold $V^1 \times V^3 = S^1 \times S^3$ equipped with the product metric $g^1 + g^3$ has $\dim_{\mathbb{H}} \ker D_{g^1+g^3} = 2$, see [7, Rem. 4, p. 12]. Rescaling this metric we can assume that the interval $(-3L, 3L)$ contains no further eigenvalues. Performing surgery on the circle $S := S^1 \times \{\text{pt.}\} \subset \mathrm{S}^1 \times \mathrm{S}^3$ gives the sphere S^4. Since S has codimension 3 the surgery result in Theorem 6 (applied with Λ large and ϵ small) tells us that there is a metric on S^4 for which the Dirac operator has eigenspaces of total dimension 2 with eigenvalues between $-l$ and l and no further eigenvalues in the interval $(-2L, 2L)$. Since the Dirac spectrum is symmetric in dimension 4 the small eigenvalues must be either $\pm l'$, $|l'| \leq |l|$, with multiplicity 1, or 0 with multiplicity 2. If we happen to get 0 as an eigenvalue we can according to [11, Thm. 1.3] make a C^2-small perturbation of the metric so as to get an arbitrarily small non-zero pair $\pm l'$ as the first eigenvalues of the Dirac

operator while the remaining eigenvalues are outside $(-L, L)$. (Note that the Dirac eigenvalues depend continuously on the metric in the C^1-topology [1, Prop. 7.1].) By rescaling this metric we get a metric $g^{l,L}$ on S^4 with the required properties.

For the rest of the proof we assume $n \geq 5$.

Suppose $n \equiv 3, 7 \pmod 8$. From Lemma 8 we know that S^n is spin bordant to a manifold (W^n, h^n) with $\dim \ker D_{h^n} = 1$. By rescaling h^n we can assume that the interval $(-3L, 3L)$ contains no non-zero eigenvalues of D_{h^n}. Since the sphere is simply connected and $n \geq 5$ there is a sequence of surgeries of codimension at least three on W^n which will produce the sphere, see [6, Proof of Thm. B]. Successive applications of Theorem 6 (with Λ large and ϵ small enough) gives a metric on S^n with one simple eigenvalue close to 0 (smaller than $|l|$), and no further eigenvalues in $(-2L, 2L)$. If we happen to get a zero eigenvalue then we take a C^2-nearby metric with no zero eigenvalue (in this case referring to [2, Thm. 3.10]). Rescaling the metric (and possibly changing orientation) gives us the required metric $g^{l,L}$.

Finally suppose $n \not\equiv 3, 7 \pmod 8$. The same argument as in the previous case gives us a metric on S^n for which the Dirac operator has eigenspaces of total dimension 2 with eigenvalues between $-l$ and l and no further eigenvalues in the interval $(-2L, 2L)$. Since the Dirac spectrum is symmetric in dimensions $n \not\equiv 3, 7 \pmod 8$ the small eigenvalues must be either $\pm l'$, $|l'| \leq |l|$, with multiplicity 1, or 0 with multiplicity 2. If we get 0 as an eigenvalue we perturb the metric to have an arbitrarily small pair $\pm l'$ as the first eigenvalues of the Dirac operator while the remaining eigenvalues are outside $(-L, L)$. By rescaling this metric we get a metric $g^{l,L}$ on S^n with the required properties. □

4. Proof of the main theorem

We can now prove Theorem 3.

Theorem 3. *Let M be a compact spin manifold of dimension $n \geq 3$ and let $L > 0$ be a real number.*

- *Suppose that $n \equiv 3, 7 \pmod 8$ and let $l_1, l_2, \ldots, l_m$ be non-zero real numbers such that $-L < l_1 < l_2 < \cdots < l_m < L$. Then there is a Riemannian metric g on M such that $\mathrm{spec}\,(\mathrm{D_g}) \cap ((-\mathrm{L}, \mathrm{L}) \setminus \{0\})$ consists precisely of $l_1, l_2, \ldots, l_m$ as simple eigenvalues.*
- *Suppose that $n \not\equiv 3, 7 \pmod 8$ and let $l_1, l_2, \ldots, l_m$ be real numbers such that $0 < l_1 < l_2 < \cdots < l_m < L$. Then there is a Riemannian metric g on M such that $\mathrm{spec}\,(\mathrm{D_g}) \cap ((-\mathrm{L}, \mathrm{L}) \setminus \{0\})$ consists precisely of $\pm l_1, \pm l_2, \ldots, \pm l_m$ as simple eigenvalues.*

Proof. Start with a metric on M for which the dimension of the kernel of the Dirac operator is minimal among all Riemannian metrics on M. A generic choice of metric will have this property, see the discussion in [2, Sec. 3]. Rescale this metric to get a metric g_0 which has no non-zero eigenvalues in the interval $(-3L, 3L)$.

For each $i = 1, \dots, m$ take a metric $g^{l_i,3L}$ on S^n as provided by Proposition 7. Rescale these metrics as $t_i g^{l_i,3L}$, where the t_i are real-valued parameters close to 1. This gives metrics with one eigenvalue $\lambda_i(t_i)$ being a monotone function of t_i passing through the value l_i when $t_i = 1$ and no further eigenvalues in $(-2L, 2L)$. The non-zero eigenvalues in $(-2L, 2L)$ of the disjoint union

$$(M, g_0) \sqcup (S^n, t_1 g^{l_1,3L}) \sqcup \cdots \sqcup (S^n, t_m g^{l_m,3L}) \tag{2}$$

are then precisely $\lambda_1(t_1), \dots, \lambda_m(t_m)$. Define the map $F : (t_1, \dots, t_m) \mapsto (\lambda_1(t_1), \dots, \lambda_m(t_m))$. Since the functions $t_i \mapsto \lambda_i(t_i)$ are monotone for t_i near 1 the map F restricted to a sphere $S = \partial B$ bounding a small ball $B \subset \mathbb{R}^m$ containing $(1, \dots, 1)$ has degree ± 1 as a map to $\mathbb{R}^m \setminus (l_1, \dots, l_m)$.

Theorem 6 (with sufficiently small ϵ and sufficiently large Λ) tells us that there is a family of metrics parametrized by B on the connected sum

$$M \# \underbrace{S^n \# \dots \# S^n}_{m \text{ copies}} = M$$

whose Dirac operators are (Λ, ϵ)-spectral close to the Dirac operators on the disjoint union (2) for $(t_1, \dots, t_m) \in B$. Since the zero eigenvalue of D_{g_0} was assumed to have minimal multiplicity the zero eigenvalue on the connected sum must have the same minimal multiplicity. Thus the summand (M, g_0) cannot contribute any small non-zero eigenvalue to the connected sum metric.

We choose B and ϵ small enough so that the non-zero eigenvalues $\tilde{\lambda}_i(t_1, \dots, t_m)$ are simple and consistently numbered for $(t_1, \dots, t_m) \in B$. We then get a continuous map $\tilde{F} : (t_1, \dots, t_m) \mapsto (\tilde{\lambda}_1, \dots, \tilde{\lambda}_m)$ which just as F has degree ± 1 as a map from S to $\mathbb{R}^m \setminus (l_1, \dots, l_m)$. It follows that $\tilde{F}$ must map some point in B to $(l_1, \dots, l_m)$, so we get a metric with the prescibed eigenvalues in $(-L, L) \setminus \{0\}$. $\square$

If Conjecture 1 is true for a manifold of dimension ≥ 3 we can choose the initial metric to have the multiplicity of the zero eigenvalue given by the index theorem, and then prescribe arbitrarily a finite number of non-zero eigenvalues of multiplicity 1.

It seems reasonable to conjecture that one can prescribe finitely many Dirac eigenvalues with arbitrary finite multiplicities. In short one could state the following conjecture.

Conjecture 9. *Except for the algebraic constraints on the spinor bundle (giving quaternionic eigenspaces and symmetric spectrum) and the constraints on the zero eigenvalue coming from the Atiyah-Singer index theorem it is possible to find a Riemannian metric on any compact spin manifold with a finite part of its Dirac spectrum arbitrarily prescribed.*

The reason why the method of this paper does not work for prescribing a multiple eigenvalue is that there is then no well-defined numbering for the eigenvalue functions as functions on the parameter space.

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